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A Problem in Potential Theory and Zero Asymptotics of Krawtchouk Polynomials

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In this paper we investigate the asymptotics of the zeros of normalized Krawtchouk polynomials $k_n(Nx, p, N)$ when the ratio of the parameters $n/N \to \alpha$ as $n, N \to \infty$. For this purpose we consider in detail a particular constrained energy problem on the interval $[0, 1]$ in the presence of an external field. We find the support and the density of the constrained extremal measure for all possible values of the parameter α. © 2000 Academic Press

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

In [4] we investigated the constrained energy problem (denoted briefly by CEP) for logarithmic potentials and derived several theoretical results. As an application we showed that the weak* limit of the normalized zero counting measures of the Krawtchouk polynomials is the solution to a particular CEP. In the case when the parameter $p = 1/2$ the solution was found explicitly. Our goal here is to settle the general case. The results of this

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paper also provide a concrete example for the solution to an inverse problem for the Toda lattice having Hamiltonian

$$H=\frac{1}{2}\sum_{k=1}^{N} y_k^2+\sum_{k=1}^{N-1} e^{x_k-x_{k+1}}.$$

For details see the paper by Deift and McLaughlin [2]. Their parameters are related to those of this paper as follows; $x \leftrightarrow \alpha$, $\lambda \leftrightarrow x$, and $t \leftrightarrow \log(p/(1-p))$.

With the notation of [11, Sect. 2.82], for given p and N the Krawtchouk polynomial of degree $n(\leqslant N)$ is given by

$$k_n(x, p, N)=\binom{N}{n}^{-1/2}(pq)^{-n/2}\sum_{s=0}^{n}(-1)^{n-s}\binom{N-x}{n-s}\binom{x}{s}p^{n-s}q^s, \tag{1.1}$$

where $p, q>0$, $p+q=1$, and $N\in\mathbf{N}$. These polynomials are orthonormal with respect to

$$\sum_{i=0}^{N}\binom{N}{i}p^iq^{N-i}\,\delta(i),$$

where $\delta(i)$ is the unit measure with mass point at i, $i=0, 1, ..., N$. As is well known the zeros of k_n are all simple and lie in the interval $(0, N)$ (see [1, 11]). They are also separated by the mass points of the measure of orthogonality. We shall convert the problem to the interval $[0, 1]$ by scaling the polynomials. Let $P_n(x)=P_n(x, p, N):=A_{p,n,N}k_n(Nx, p, N)$, where the factor

$$A_{p,n,N}=\binom{N}{n}^{1/2}(pq)^{n/2}\,n!\,N^{-n} \tag{1.2}$$

is chosen so that P_n has leading coefficient one. Then the polynomials $P_n(x)$ satisfy the orthogonality relation

$$\sum_{i=0}^{N}\binom{N}{i}p^iq^{N-i}P_n\left(\frac{i}{N}\right)P_m\left(\frac{i}{N}\right)=A_{p,n,N}^2\,\delta_{n,m},\qquad n, m=0, 1, ..., N. \tag{1.3}$$

Also P_n is the discrete L_2-minimal monic polynomial, i.e., if we denote by $\tau_N:=\{i/N: i=0, ..., N\}$ the $N+1$ equally spaced points of $[0, 1]$, then

$$\|P_n\|_{\tau_N}^2=\min_{p_n=x^n+\cdots}\left\{\sum_{i=0}^{N}\binom{N}{i}p^iq^{N-i}p_n^2\left(\frac{i}{N}\right)\right\}=A_{p,n,N}^2,\qquad n=0, 1, ..., N, \tag{1.4}$$

where the norm $\|p_n\|_{\tau N}$ of a polynomial p_n is defined by the square root of the sum above.

Next we introduce the terminology and the notations that are required to state our main results. Let χ_{P_n} be the *normalized zero counting measure* of the polynomial P_n, i.e.,

$$\chi_{P_n} := \frac{1}{n} \sum_{P_n(z)=0} \delta(z),$$

where $\delta(z)$ is the measure with unit mass at z. Set $\sigma = \sigma_\alpha := (1/\alpha)\, m$, where m is the Lebesgue measure on $[0, 1]$. We consider the weight function $w := \exp(-Q_{\alpha, p})$ on $[0, 1]$, where

$$Q_{\alpha, p}(x) := \frac{1}{2\alpha} \{x \log x + (1-x) \log(1-x) - x \log p - (1-x) \log(1-p)\}. \tag{1.5}$$

The *weighted energy* of a probability measure μ with $S_\mu := \operatorname{supp}(\mu) \subset [0, 1]$ is given by

$$I_w(\mu) := \int\int \log \frac{1}{|x-t|\, w(x)\, w(t)}\, d\mu(x)\, d\mu(t)$$

and its *logarithmic potential* is given by

$$U^\mu(x) := \int \log \frac{1}{|x-t|}\, d\mu(t).$$

We define the *σ-constrained extremal measure* λ_w^σ to be the unique solution of the minimal energy problem

$$I_w(\lambda_w^\sigma) = \min\{I_w(\mu) : \mu \in \mathcal{M}^\sigma\}, \tag{1.6}$$

where $\mathcal{M}^\sigma := \{\mu : \|\mu\| = 1 \text{ and } 0 \leqslant \mu \leqslant \sigma\}$ (see [9, 4]). The notation $\mu \leqslant \sigma$ means that $\sigma - \mu$ is a positive measure. We shall denote the constrained extremal measure by $\lambda_{\alpha, p}$ in order to emphasize the dependence on the parameters. We also use $\mu_{\alpha, p}$ to denote the solution of the unconstrained problem (i.e., when the minimum in (1.6) is taken over all probability measures μ with $S_\mu \subset [0, 1]$).

The following theorem about the weak* limit of χ_{P_n} was proved in [4].

THEOREM A. *Let $k_n(x, p, N)$ be the Krawtchouk polynomial* (1.1) *and $P_n(x) := A_{p, n, N} k_n(Nx, p, N)$ be the associated normalized monic polynomials. Suppose $N = N_j$, $n = n_j$ are sequences satisfying $N_j \to \infty$, $n_j \to \infty$, and*

$n_j/N_j \to \alpha < 1$ as $j \to \infty$. Then the normalized zero counting measures of P_n and the nth root of the discrete norms satisfy

$$\chi_{P_n} \xrightarrow{*} \lambda_{\alpha, p} \qquad \text{as} \quad j \to \infty, \tag{1.7}$$

and

$$\lim_{j \to \infty} \|P_n\|_{\tau_N}^{1/n} = \frac{\sqrt{\alpha\beta pq}}{e\beta^{1/(2\alpha)}}, \tag{1.8}$$

where $\beta = 1 - \alpha$, $q = 1 - p$, and

$$w(x) = \exp(-Q_{\alpha, p}(x)) = [x^x(1-x)^{1-x}/p^x(1-p)^{1-x}]^{-1/(2\alpha)}.$$

While the nth root asymptotics for the norms in (1.8) can be found easily from (1.2) (see [4, Remark 3.6]), the zero distribution requires a much deeper analysis. The case $p = q = 1/2$ is relatively easy and was handled by the authors in [4]. Here we investigate the general case. The determination of the support of the limiting measure $\lambda_{\alpha, p}$ is given in Theorem 2 below; its distribution is derived in Theorem 3. Some properties of the class of measures $\lambda_{\alpha, p}$ and $\mu_{\alpha, p}$ are summarized in the next theorem. For simplicity we shall denote hereafter $\beta := 1 - \alpha$ and $q := 1 - p$.

THEOREM 1. *Let $0 < \alpha, p < 1$ and set $\beta = 1 - \alpha$ and $q = 1 - p$. Then the following hold*:

(a) *$\alpha\lambda_{\alpha, p} + \beta\lambda_{\beta, q} = m$, where m is the Lebesgue measure on $[0, 1]$.*

(b) *$(d\lambda_{\alpha, p}/dx)(1-x) = (d\lambda_{\alpha, q}/dx)(x)$ and $(d\mu_{\alpha, p}/dx)(1-x) = (d\mu_{\alpha, q}/dx)(x)$.*

(c) *$(\alpha\mu_{\alpha, p} + \beta\mu_{\beta, q})|_{S_{\mu_{\alpha, p}} \cap S_{\mu_{\beta, q}}} \leqslant m$.*

According to this theorem the only case that we need to consider is when $0 \leqslant \alpha \leqslant 1/2$ and $p < 1/2$ (we recall that the case $p = 1/2$ has already been considered in [4]).

As in the unconstrained case, the most difficult part of the analysis is the determination of the support of the extremal measure. The next theorem deals simultaneously with the supports of the constrained extremal measure $\lambda_{\alpha, p}$ and its dual $\lambda_{1-\alpha, 1-p}$.

THEOREM 2. *Let $0 < \alpha < 1$ and $0 < p < 1/2$. Define the constants $A = A_{\alpha, p}$ and $B = B_{\alpha, p}$ by the formulas*

$$A := \alpha q + \beta p - 2\sqrt{\alpha\beta pq}, \qquad B := \alpha q + \beta p + 2\sqrt{\alpha\beta pq}, \tag{1.9}$$

where $\beta = 1 - \alpha$ and $q = 1 - p$.

(a) *If* $0<\alpha<p$, *then* $S_{\lambda_{\alpha,p}}=[A,B]$ *and* $S_{\lambda_{\beta,q}}=[0,1]$.

(b) *If* $p\leqslant\alpha<1-p$, *then* $S_{\lambda_{\alpha,p}}=[0,B]$ *and* $S_{\lambda_{\beta,q}}=[A,1]$.

(c) *If* $1-p\leqslant\alpha<1$, *then* $S_{\lambda_{\alpha,p}}=[0,1]$ *and* $S_{\lambda_{\beta,q}}=[A,B]$.

The density of the constrained extremal measure is described in the next result.

THEOREM 3. *With the notation of the previous theorem, the density of the* (α, p)*-constrained extremal measure* $\lambda_{\alpha,p}$ *is given as follows.*

(a) *If* $0<\alpha<p$,

$$\frac{d\lambda_{\alpha,p}}{dt}=\frac{1}{\alpha\pi}\left\{\frac{\pi}{2}-\arctan\sqrt{\frac{A(B-t)}{B(t-A)}}-\arctan\sqrt{\frac{(1-B)(t-A)}{(1-A)(B-t)}}\right\} \tag{1.10}$$

for $t\in[A,B]$ *and* $d\lambda_{\alpha,p}/dt=0$ *otherwise.*

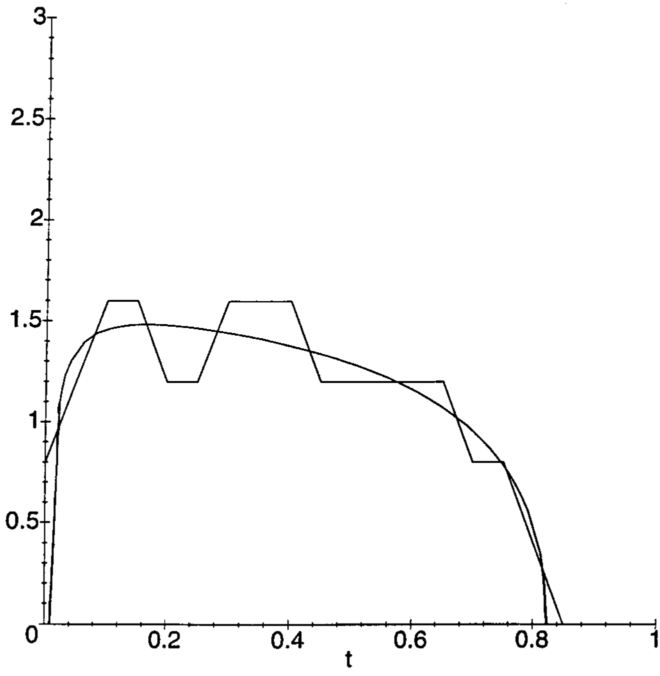

FIG. 1. Case $N=101$, $n=25$, $p=1/3$, $\alpha=0.25$, $A=0.0084$, $B=0.8249$.

(b) *If* $p \leqslant \alpha < 1 - p$,

$$\frac{d\lambda_{\alpha,\,p}}{dt} = \frac{1}{\alpha\pi}\left\{\frac{\pi}{2} + \arctan\sqrt{\frac{A(B-t)}{B(t-A)}} - \arctan\sqrt{\frac{(1-B)(t-A)}{(1-A)(B-t)}}\right\}, \tag{1.11}$$

for $t \in [A, B]$, $d\lambda_{\alpha,\,p}/dt = 1/\alpha$ *on* $[0, A]$, *and zero elsewhere.*

(c) *If* $1 - p \leqslant \alpha < 1$,

$$\frac{d\lambda_{\alpha,\,p}}{dt} = \frac{1}{\alpha\pi}\left\{\frac{\pi}{2} + \arctan\sqrt{\frac{A(B-t)}{B(t-A)}} + \arctan\sqrt{\frac{(1-B)(t-A)}{(1-A)(B-t)}}\right\}, \tag{1.12}$$

for $t \in [A, B]$ *and* $d\lambda_{\alpha,\,p}/dt = 1/\alpha$ *on* $[0, A] \cup [B, 1]$.

The results of Theorems 2 and 3 are graphically illustrated in Figs. 1, 2, and 3, where we compare the (polygonal line) density of the zeros of the Krawtchouk polynomials with the density of the limiting measures for different values of the parameter α. Notice how the shape of the density of the extremal measure changes as we pass through the critical values $\alpha = p$ and $\alpha = 1 - p$.

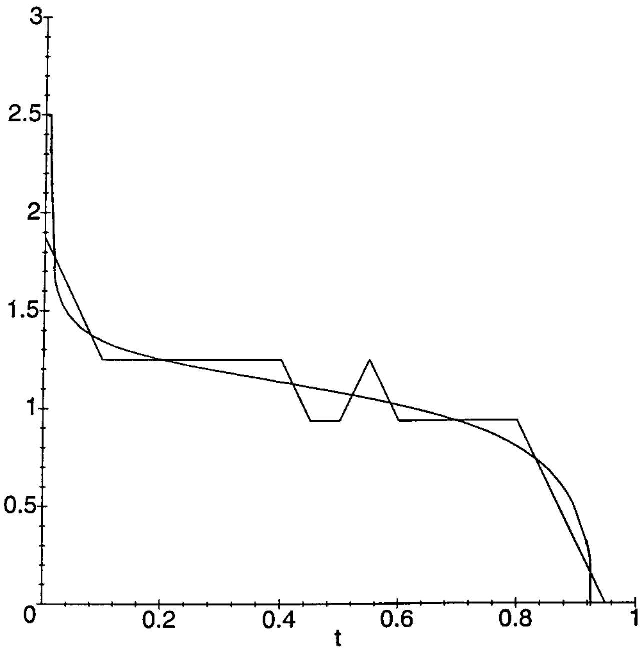

FIG. 2. Case $N = 81$, $n = 32$, $p = 1/3$, $\alpha = 0.4$, $A = 0.0048$, $B = 0.9285$.

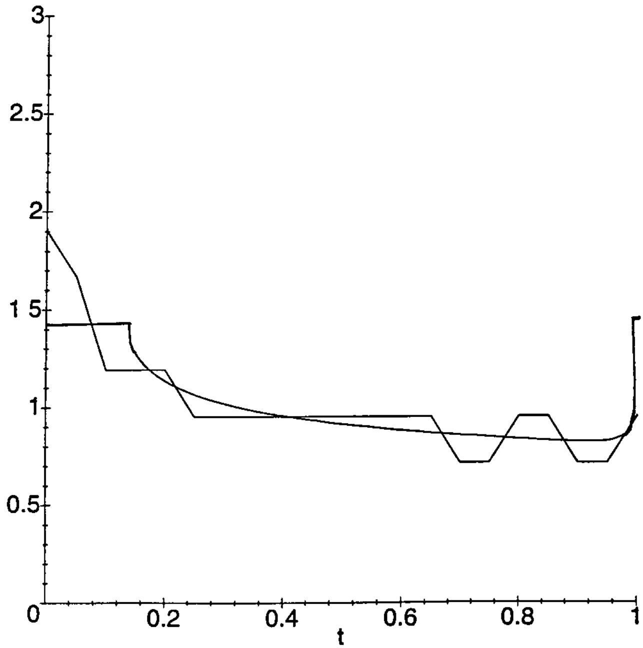

FIG. 3. Case $N=61$, $n=42$, $p=1/3$, $\alpha=0.7$, $A=0.1346$, $B=0.9987$.

The computations for the zero densities of the normalized Krawtchouk polynomials were done with MAPLE in accordance with the following algorithm:

1. Divide $[0, 1]$ into l equal subintervals by the points $x_i = i/l$, $i=0, ..., l$.

2. For fixed j define the value of the density of the zeros $P_n(x)$ at x_i to be

$$\psi_{N,n}(x_i) := \frac{|\{z \in [x_{i-j}, x_{i+j}] \mid P_n(z)=0\}|}{n\,|x_{i+j} - x_{i-j}|},$$

where $x_{-j} = \cdots = x_{-1} = 0$ and $x_{l+1} = \cdots = x_{l+j} = 1$.

3. Define $\psi_{N,n}(x)$ to be the piecewise linear function determined by the points $\{(x_i, \psi_{N,n}(x_i))\}_{i=0}^{l}$.

In Figs. 1–3, we have taken $l=20$, $j=1$, and plotted the density $\psi_{N,n}$ of the zeros of the Krawtchouk polynomials versus the density of the extremal measure $d\lambda_{\alpha,p}/dt$.

2. PROOF OF THEOREM 1

Proof. (a) We note first that, for $x \in [0, 1]$,

$$U^m(x) = \int_0^1 \log \frac{1}{|x-t|}\, dt = -x \log x - (1-x) \log(1-x) + 1, \quad (2.1)$$

and therefore,

$$Q_{\alpha, p}(x) = -\frac{1}{2\alpha} U^m(x) + \frac{1}{2\alpha} x \log \frac{q}{p} + \frac{1}{2\alpha} (1 - \log q). \quad (2.2)$$

From [4, Theorem 2.1 and Remark 2.3] the variational inequalities for the logarithmic potential of $\lambda_{\alpha, p}$ are

$$\begin{aligned} U^{\lambda_{\alpha, p}}(x) + Q_{\alpha, p}(x) &\geqslant C_{\alpha, p} \quad \text{on} \quad S_{\sigma - \lambda_{\alpha, p}} \\ U^{\lambda_{\alpha, p}}(x) + Q_{\alpha, p}(x) &\leqslant C_{\alpha, p} \quad \text{on} \quad S_{\lambda_{\alpha, p}}, \end{aligned} \quad (2.3)$$

where $\sigma = (1/\alpha)\, m$ and $C_{\alpha, p}$ is a constant. Using the formula

$$\alpha Q_{\alpha, p}(x) + \beta Q_{\beta, q}(x) = -U^m(x) + 1 - \log \sqrt{pq}, \quad (2.4)$$

we find that the dual measure $\nu := (\sigma - \lambda_{\alpha, p}) \cdot (\alpha/\beta)$ satisfies the inequalities

$$\begin{aligned} U^{\nu}(x) = Q_{\beta, q}(x) &\geqslant -\frac{\alpha}{\beta} C_{\alpha, p} + \frac{1}{\beta} - \frac{1}{2\beta} \log pq \quad \text{on} \quad S_{\sigma_\beta - \nu} \\ U^{\nu}(x) = Q_{\beta, q}(x) &\leqslant -\frac{\alpha}{\beta} C_{\alpha, p} + \frac{1}{\beta} - \frac{1}{2\beta} \log pq \quad \text{on} \quad S_{\nu}, \end{aligned} \quad (2.5)$$

where $\sigma_\beta = (1/\beta)\, m$. Moreover, $\nu \leqslant \sigma_\beta$ and $\|\nu\| = 1$, so by the uniqueness of the σ_β-constrained extremal measure $\nu = \lambda_{\beta, q}$. Thus we obtain

$$\alpha \lambda_{\alpha, p} + \beta \lambda_{\beta, q} = m.$$

(b) The proof of this part is an easy consequence of the fact that

$$Q_{\alpha, p}(1-x) = Q_{\alpha, p}(x)$$

and the uniqueness of the extremal measure.

(c) From [10, Theorem 1.3], since $[0, 1]$ is regular set w.r.t. the Dirichlet problem, we can write the equilibrium inequalities for the potentials of $\mu_{\alpha, p}$ and $\mu_{\beta, q}$ as

$$U^{\mu_{\alpha, p}}(x) + Q_{\alpha, p}(x) \geqslant F_{\alpha, p} \quad \text{on} \quad [0, 1] \quad (2.6)$$

$$U^{\mu_{\alpha, p}}(x) + Q_{\alpha, p}(x) \leqslant F_{\alpha, p} \quad \text{on} \quad S_{\mu_{\alpha, p}}, \quad (2.7)$$

and

$$U^{\mu_{\beta,q}}(x)+Q_{\beta,q}(x)\geqslant F_{\beta,q} \qquad \text{on} \qquad [0,1] \tag{2.8}$$

$$U^{\mu_{\beta,q}}(x)+Q_{\beta,q}(x)\leqslant F_{\beta,q} \qquad \text{on} \qquad S_{\mu_{\beta,q}}. \tag{2.9}$$

Adding α times (2.6) to β times (2.8) and using (2.4) we obtain

$$U^{\alpha\mu_{\alpha,p}+\beta\mu_{\beta,q}}(x)-U^{m}(x)\geqslant \alpha F_{\alpha,p}+\beta F_{\beta,q}+\log\sqrt{pq}-1 \qquad \text{on} \qquad [0,1].$$

By the principle of domination this inequality can be extended to hold everywhere (see [7, Theorem 1.27]). On the intersection $S_{\mu_{\alpha,p}}\cap S_{\mu_{\beta,q}}$ the opposite inequality holds (just by adding (2.7) and (2.9)) and by [10, Theorem IV.4.5] we obtain the assertion of part (c). ∎

3. THE SUPPORT OF $\lambda_{\alpha,p}$

To find the σ-constrained extremal measure $\lambda_{\alpha,p}$ it is essential to determine the support of this measure.

First observe that the corresponding external field $Q_{\alpha,p}$ is a convex function on $[0,1]$, which according to Theorem 2.16 in [4] implies that $S_{\lambda_{\alpha,p}}$ is an interval.

Case 1. $\alpha<p$. In this case we focus first on the solution $\mu_{\alpha,p}$ of the unconstrained weighted energy problem on the interval $[0,1]$ with weight function $w_{\alpha,p}=\exp(-Q_{\alpha,p})$ (cf. [10]). One reason for considering the unconstrained problem is that there is a relationship between the support of $\lambda_{\alpha,p}$ and $\mu_{\alpha,p}$; namely that $S_{\mu_{\alpha,p}}\subset S_{\lambda_{\alpha,p}}$ (see [4, Theorem 2.6]). The following lemma determines the support of $\mu_{\alpha,p}$. In Theorem 3(a) we determine the density of this measure when $\alpha<p$, from which we deduce that $\mu_{\alpha,p}\leqslant\sigma$, and therefore $\lambda_{\alpha,p}=\mu_{\alpha,p}$, which settles this case.

LEMMA 4. *Let $p<1/2$ and $w=\exp(-Q_{\alpha,p})$ be a weight function on $[0,1]$, where $Q_{\alpha,p}$ is given in* (1.5). *Then for the support $S_{\mu_{\alpha,p}}$ of the weighted equilibrium measure $\mu_{\alpha,p}$ we have*

(a) *If $0<\alpha<p$, then $S_{\mu_{\alpha,p}}=[A,B]$ where A, B are given in* (1.9).

(b) *If $p\leqslant\alpha<\alpha_p:=(1/2)(\log(\sqrt{1-p}/\sqrt{p})+1)$, then $S_{\mu_{\alpha,p}}=[0,b_\alpha]$, where b_α is the unique solution of the equation*

$$\frac{1}{b}-\frac{1}{2\alpha}\left\{\log\frac{\sqrt{b}\,\sqrt{1-p}}{(1+\sqrt{1-b})\,\sqrt{p}}+\frac{1}{1+\sqrt{1-b}}\right\}=0.$$

(c) *If $\alpha_p\leqslant\alpha<1$, then $S_{\mu_{\alpha,p}}=[0,1]$.*

Remark. Comparing Theorem 2 and Lemma 4 we observe an interesting phenomenon. Let α be gradually increasing from 0 to 1 with p fixed, $p<1/2$. Then the supports $S_{\mu_{\alpha,p}}$ and $S_{\lambda_{\alpha,p}}$ are intervals expanding with α (cf. [10, Theorem IV.1.6 (f)]). At first, when $0<\alpha\leqslant p$, the two measures coincide (and the supports are the same), i.e., the constraint is not active. When $p<\alpha<1-p$, the constraint becomes active and the two measures are different. We have that $S_{\mu_{\alpha,p}}=[0,b_\alpha]$ and $S_{\lambda_{\alpha,p}}=[0,B_{\alpha,p}]$ with $b_\alpha<B_{\alpha,p}$. When $1-p\leqslant\alpha<\alpha_p$, still $S_{\mu_{\alpha,p}}=[0,b_\alpha]$ $(b_\alpha<1)$, while $S_{\lambda_{\alpha,p}}=[0,1]$. Finally, when $\alpha_p\leqslant\alpha<1$, $S_{\mu_{\alpha,p}}=S_{\lambda_{\alpha,p}}=[0,1]$.

Proof. We shall use as a basic tool the Mhaskar–Saff functional, defined for any compact set K

$$F(K)=\log\operatorname{cap}(K)-\int Q\,d\omega_K,$$

where ω_K is the (unweighted) equilibrium measure of K and Q is the external field of the corresponding energy problem (see [8; 10, Theorem IV.1.5]). Since $Q_{\alpha,p}$ is convex, the support of $\mu_{\alpha,p}$ is an interval (cf. [10, Theorem IV.1.11]), say $[a,b]\subset[0,1]$, where the pair $\{a,b\}$ maximizes

$$F([a,b])=\log\operatorname{cap}([a,b])-\int Q_{\alpha,p}\,d\omega_{[a,b]} \tag{3.1}$$

for $0\leqslant a<b\leqslant 1$.

It follows from the formula for $Q_{\alpha,p}$ in (1.5) that for fixed p, the support $S_{\mu_{\alpha,p}}$ is an increasing function of α (cf. [10, Theorem IV.1.6 (f)]), i.e., we have a family of intervals expanding with α.

Next we compute the F-functional. After the change of variables

$$x=\frac{b+a}{2}-\frac{b-a}{2}\cos\theta,$$

we obtain

$$\begin{aligned}F([a,b])&=\log\frac{b-a}{4}-\frac{1}{\pi}\int_0^\pi Q_{\alpha,p}\left(\frac{b+a}{2}-\frac{b-a}{2}\cos\theta\right)d\theta\\&=\log\frac{b-a}{4}-\frac{1}{2\alpha}\{I_1+I_2+I_3-\log q\},\end{aligned} \tag{3.2}$$

where

$$I_1 := \frac{1}{\pi}\int_0^{\pi}\left(\frac{b+a}{2}-\frac{b-a}{2}\cos\theta\right)\log\left(\frac{b+a}{2}-\frac{b-a}{2}\cos\theta\right)d\theta,$$

$$I_2 := \frac{1}{\pi}\int_0^{\pi}\left(\frac{2-a-b}{2}+\frac{b-a}{2}\cos\theta\right)\log\left(\frac{2-a-b}{2}+\frac{b-a}{2}\cos\theta\right)d\theta, \quad (3.3)$$

$$I_3 := \frac{1}{\pi}\log\frac{q}{p}\int_0^{\pi}\left(\frac{b+a}{2}-\frac{b-a}{2}\cos\theta\right)d\theta=\frac{b+a}{2}\log\frac{q}{p}.$$

From standard formulas (see, e.g., [10, Lemma IV.1.15]) we obtain

$$\begin{aligned} J_1 &:= \frac{b+a}{2}\frac{1}{\pi}\int_0^{\pi}\log\left(\frac{b+a}{2}-\frac{b-a}{2}\cos\theta\right)d\theta \\ &= (b+a)\log\left(\frac{\sqrt{b}+\sqrt{a}}{2}\right), \end{aligned} \quad (3.4)$$

and with the help of integration by parts we find

$$\begin{aligned} J_2 &:= \frac{b-a}{2}\left\{-\frac{1}{\pi}\int_0^{\pi}\log\left(\frac{b+a}{2}-\frac{b-a}{2}\cos\theta\right)d\sin\theta\right\} \\ &= \frac{b-a}{2}\cdot\left\{\frac{b+a}{b-a}-\frac{4ab}{(b-a)^2}\cdot\frac{b-a}{2\sqrt{ab}}\right\}=\frac{b+a}{2}-\sqrt{ab}. \end{aligned} \quad (3.5)$$

Thus, combining (3.4) and (3.5) we have

$$I_1 = J_1 + J_2 = (b+a)\log\left(\frac{\sqrt{b}+\sqrt{a}}{2}\right)+\frac{b+a}{2}-\sqrt{ab}. \quad (3.6)$$

In a similar fashion we compute

$$I_2 = (2-a-b)\log\left(\frac{\sqrt{1-a}+\sqrt{1-b}}{2}\right)+\frac{2-a-b}{2}-\sqrt{(1-a)(1-b)}. \quad (3.7)$$

Combining (3.3), (3.6), and (3.7), the Mhaskar–Saff functional can be written as

$$\begin{aligned} F([a,b]) = \log\frac{b-a}{4}-\frac{1}{2\alpha}\Big\{&(b+a)\log(\sqrt{a}+\sqrt{b})-\log 4q+1-\sqrt{ab} \\ &+(2-a-b)\log(\sqrt{1-a}+\sqrt{1-b}) \\ &-\sqrt{(1-a)(1-b)}+\frac{b+a}{2}\log\frac{q}{p}\Big\}. \end{aligned} \quad (3.8)$$

The partial derivatives of F with respect to a and b are given by

$$\frac{\partial F}{\partial a} = -\frac{1}{b-a} - \frac{1}{2\alpha}\left\{\log\frac{(\sqrt{a}+\sqrt{b})\sqrt{q}}{(\sqrt{1-a}+\sqrt{1-b})\sqrt{p}} - \frac{1}{2}\left(\frac{\sqrt{b}-\sqrt{a}}{\sqrt{a}+\sqrt{b}} + \frac{\sqrt{1-a}-\sqrt{1-b}}{\sqrt{1-a}+\sqrt{1-b}}\right)\right\},$$

and

$$\frac{\partial F}{\partial b} = \frac{1}{b-a} - \frac{1}{2\alpha}\left\{\log\frac{(\sqrt{a}+\sqrt{b})\sqrt{q}}{(\sqrt{1-a}+\sqrt{1-b})\sqrt{p}} + \frac{1}{2}\left(\frac{\sqrt{b}-\sqrt{a}}{\sqrt{a}+\sqrt{b}} + \frac{\sqrt{1-a}-\sqrt{1-b}}{\sqrt{1-a}+\sqrt{1-b}}\right)\right\}.$$

We investigate the functional first for local extrema inside the domain $0<a<b<1$. Setting the partial derivatives to zero and by subtracting and adding the two equalities, we derive after simplification the system

$$\frac{2}{b-a} = \frac{1}{2\alpha}\left\{\frac{\sqrt{b}-\sqrt{a}}{\sqrt{a}+\sqrt{b}} + \frac{\sqrt{1-a}-\sqrt{1-b}}{\sqrt{1-a}+\sqrt{1-b}}\right\}, \tag{3.9}$$

$$0 = \frac{1}{2\alpha}\left\{2\log\frac{\sqrt{a}+\sqrt{b}}{\sqrt{1-a}+\sqrt{1-b}} + \log\frac{q}{p}\right\}. \tag{3.10}$$

Rationalizing (3.9) and multiplying by $\alpha(b-a)$ leads to

$$2\alpha = 1 - \sqrt{ab} - \sqrt{(1-a)(1-b)}. \tag{3.11}$$

Solving (3.10) we obtain

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{1-a}+\sqrt{1-b}} = \frac{\sqrt{p}}{\sqrt{q}}. \tag{3.12}$$

When we square both sides of the last equation and cross-multiply we get

$$q(a+b+2\sqrt{ab}) = p(2-a-b+2\sqrt{1-a}\sqrt{1-b}),$$

from which we find that

$$\sqrt{1-a}\sqrt{1-b} = \frac{b+a}{2p} + \frac{q\sqrt{ab}}{p} - 1.$$

Substituting back in (3.11) we find

$$4\beta p = a + b + 2\sqrt{ab} = (\sqrt{a} + \sqrt{b})^2. \tag{3.13}$$

Next, from (3.11) we get

$$-a - b = -4\alpha + 4\alpha^2 - 2\sqrt{ab} + 4\alpha\sqrt{ab}$$

and taking into account that $a + b = 4\beta p - 2\sqrt{ab}$, $\alpha + \beta = 1$, and $p + q = 1$ we find

$$-4p\beta + 2\sqrt{ab} = -4\alpha\beta - 2\sqrt{ab} + 4\alpha\sqrt{ab}$$

or

$$\sqrt{ab} = p - \alpha. \tag{3.14}$$

The last equation shows that a solution to the system (3.9) and (3.10) exists if and only if $\alpha < p$. In this case, from (3.13) we have that $\sqrt{a} = 2\sqrt{\beta p} - \sqrt{b}$, so we substitute in (3.14) to get a quadratic equation for $\sqrt{b}$. There are two positive solutions (here we use the fact that $\alpha < p$) given by

$$\sqrt{b} = \sqrt{\beta p} \pm \sqrt{\beta p - p + \alpha} = \sqrt{\beta p} \pm \sqrt{\alpha q}.$$

Since $\sqrt{b} > \sqrt{a}$ we finally find

$$\sqrt{b} = \sqrt{\beta p} + \sqrt{\alpha q},$$
$$\sqrt{a} = \sqrt{\beta p} - \sqrt{\alpha q},$$

which gives the formulas (1.9).

We now investigate the functional (3.8) on the boundary. Since for $a = b$ the functional is $-\infty$, we have to consider the cases when $a = 0$ and when $b = 1$. It is easy to show that if $b = 1$, in order to achieve a maximum of the functional (3.8), we must have $a = 0$ (see [3] for details). So let $a = 0$. Then

$$f_\alpha(b) := \frac{\partial F}{\partial b}(0, b) = \frac{1}{b} - \frac{1}{2\alpha}\left\{\log\frac{\sqrt{b}\sqrt{q}}{(1+\sqrt{1-b})\sqrt{p}} + \frac{1}{1+\sqrt{1-b}}\right\}, \tag{3.15}$$

is decreasing on (0, 1) and tends to $+\infty$ as $b \to 0^+$. Let α_p be the solution of $f_\alpha(1) = 0$, i.e.,

$$\alpha_p := \frac{1}{2}\left\{\log\frac{\sqrt{q}}{\sqrt{p}} + 1\right\}. \tag{3.16}$$

It is not difficult to prove that $\alpha_p > 1 - p$. Let also b_α be the solution of the equation $f_\alpha(b) = 0$. Now we have the following:

(a) If $\alpha \leqslant p$, then $f_\alpha(1) < 0$, which means that $b_\alpha < 1$ and $F([0, b])$ is maximized at b_α. Since $(\partial F/\partial a)(0, b_\alpha) > 0$, we will increase the functional $F([\varepsilon, b_\alpha])$ for small ε. Thus the absolute maximum is not achieved on the boundary, but inside the domain, i.e., $S_{\mu_{\alpha,p}} = [A, B]$.

(b) If $p < \alpha < \alpha_p$, then we have no critical points inside the domain $0 \leqslant a < b \leqslant 1$, so the maximum is achieved on the boundary, i.e., $S_{\mu_{\alpha,p}} = [0, b_\alpha]$ with $b_\alpha < 1$, because $f_\alpha(1) < 0$ in this case.

(c) Finally, if $\alpha \geqslant \alpha_p$ then $f_\alpha(1) \geqslant 0$ and the maximum of $F([a, b])$ is achieved for $[0, 1]$, i.e., $S_{\mu_{\alpha,p}} = [0, 1]$. ∎

Case 2. $p \leqslant \alpha \leqslant 1/2$.

Lemma 5. *Let* $0 < p \leqslant \alpha \leqslant 1/2$ *and set* $\beta = 1 - \alpha, q = 1 - p$. *Then* $S_{\lambda_{\alpha,p}} = [0, B]$ *and* $S_{\lambda_{\beta,q}} = [A, 1]$, *where A and B are the same as in* (1.9).

Proof. We know that in this case $S_{\lambda_{\alpha,p}} = [0, B]$ and $S_{\lambda_{\beta,q}} = [A, 1]$ for suitable constants A and B with $0 < A < B < 1$. This is so, because $Q_{\alpha,p}$ and $Q_{\beta,q}$ are convex functions on $[0, 1]$ (and therefore the corresponding supports are intervals) and the union of the two supports is the whole interval $[0, 1]$. Moreover, $0 \in S_{\mu_{\alpha,p}} \subset S_{\lambda_{\alpha,p}}$ and $1 \in S_{\mu_{\beta,q}} \subset S_{\lambda_{\beta,q}}$ (see Theorem 1(b) and Lemma 4). That $0 < A$ and $B < 1$ can be seen from the unboundedness of the measures $\mu_{\alpha,p}$ near 0 and $\mu_{\beta,q}$ near 1 (which can be argued using [10, Theorem IV.4.9]), the saturation principle [4, Theorem 2.6], and Theorem 1(a). So what is left is to show that the constants A, B are given by (1.9). First we observe that on the interval $[0, A]$ the measure $\lambda_{\alpha,p}$ coincides with the constraint measure. Similarly, $\lambda_{\beta,q}$ coincides with σ on $[B, 1]$. Denote $m_1 := m|_{[0,A]}$, $m_2 := m|_{[A,B]}$ and $m_3 := m|_{[B,1]}$ (recall that m is the Lebesgue measure on $[0, 1]$). Then $m = m_1 + m_2 + m_3$. Define the measures

$$\lambda_1 := \left(\lambda_{\alpha,p} - \frac{m_1}{\alpha}\right)\left(1 - \frac{A}{\alpha}\right)^{-1}, \qquad \lambda_2 := \left(\lambda_{\beta,q} - \frac{m_3}{\beta}\right)\left(1 - \frac{1-B}{\beta}\right)^{-1}.$$

From [4, Corollary 2.10] we have that λ_1 and λ_2 are unconstrained extremal measures on the intervals $[A, 1]$ and $[0, B]$ for the external fields $Q_{1,A}$ and $Q_{2,B}$, respectively, where

$$\begin{aligned} Q_{1,A}(x) &:= \left(1 - \frac{A}{\alpha}\right)^{-1} \{U^{m_1/\alpha}(x) + Q_{\alpha,p}(x)\} \\ &= \frac{1}{2(\alpha - A)} \left\{U^{m_1}(x) - U^{m_2}(x) - U^{m_3}(x) + x \log\frac{q}{p} + c_1\right\}, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} Q_{2,B}(x) &:= \left(1-\frac{1-B}{\beta}\right)^{-1} \{U^{m_3/\beta}(x)+Q_{\beta,q}(x)\} \\ &= \frac{1}{2(B-\alpha)}\left\{-U^{m_1}(x)-U^{m_2}(x)+U^{m_3}(x)+x\log\frac{p}{q}+c_2\right\}. \end{aligned} \tag{3.18}$$

Here $c_1=1-\log q$ and $c_2=1-\log p$. Replace A, B by variables a, b in the definition of $Q_{1,A}$ and $Q_{2,B}$. The corresponding Mhaskar–Saff functionals are

$$\begin{aligned} F_1([a,b]) &= \log\frac{b-a}{4}-\int_a^b Q_{1,a}(x)\,d\omega_{[a,b]}, \\ F_2([a,b]) &= \log\frac{b-a}{4}-\int_a^b Q_{2,b}(x)\,d\omega_{[a,b]}, \end{aligned} \tag{3.19}$$

where $d\omega_{[a,b]}=dx/(\pi\sqrt{(x-a)(b-x)})$ is the equilibrium measure on the interval $[a,b]$. Since $\operatorname{supp}(\lambda_1)=\operatorname{supp}(\lambda_2)=[A,B]$, we have that B maximizes $F_1([A,b])$ and A maximizes $F_2([a,B])$, therefore

$$\frac{\partial F_1}{\partial b}(A,B)=0, \qquad \frac{\partial F_2}{\partial a}(A,B)=0.$$

Thus, we can write the system

$$\frac{\partial F_1}{\partial b}+\frac{\partial F_2}{\partial a}=0, \qquad \frac{\partial F_1}{\partial b}-\frac{\partial F_2}{\partial a}=0. \tag{3.20}$$

We now compute the integrals

$$f_i(a,b):=\int_a^b U^{m_i}(x)\,d\omega_{[a,b]}, \qquad i=1,2,3. \tag{3.21}$$

For this purpose we remind the reader that the potential

$$U^{\omega_{[a,b]}}(z)=-\log\left|\frac{2z-a-b+2\sqrt{(z-a)(z-b)}}{4}\right|,$$

where the branch of $\sqrt{(z-a)(z-b)}$ behaves like z at infinity. After changing the order of integration we get

$$
\begin{aligned}
f_1(a,b) &= \int_0^a U^{\omega[a,b]}(x)\,dx \\
&= \int_0^a -\log\left|\frac{2x-a-b-2\sqrt{(x-a)(x-b)}}{4}\right| dx \\
&= -x\log\left|\frac{2x-a-b-2\sqrt{(x-a)(x-b)}}{4}\right|\Bigg|_0^a \\
&\quad + \int_0^a \frac{-x+(a+b)/2-(a+b)/2}{\sqrt{(x-a)(x-b)}}\,dx \\
&= -a\log\frac{b-a}{4} - \sqrt{(x-a)(x-b)}\Big|_0^a \\
&\quad + \frac{a+b}{2}\log\left|\frac{2x-a-b-2\sqrt{(x-a)(x-b)}}{4}\right|\Bigg|_0^a \\
&= \frac{b-a}{2}\log\frac{b-a}{4} - (a+b)\log\frac{\sqrt{b}+\sqrt{a}}{2} + \sqrt{ab}.
\end{aligned}
\tag{3.22}
$$

On the interval $[a, b]$ the potential $U^{\omega[a,b]}(z) = -\log\{(b-a)/4\}$; therefore

$$
f_2(a,b) = -(b-a)\log\frac{b-a}{4}. \tag{3.23}
$$

Similar computations for $f_3(a, b)$ yield (observe the change of the sign of the square root in the log term)

$$
\begin{aligned}
f_3(a,b) &= \int_b^1 U^{\omega[a,b]}(x)\,dx \\
&= \int_b^1 -\log\left|\frac{2x-a-b+2\sqrt{(x-a)(x-b)}}{4}\right| dx \\
&= -(x-1)\log\left|\frac{2x-a-b+2\sqrt{(x-a)(x-b)}}{4}\right|\Bigg|_b^1 \\
&\quad + \int_b^1 \frac{x-1+(a+b)/2-(a+b)/2}{\sqrt{(x-a)(x-b)}}\,dx \\
&= (b-1)\log\frac{b-a}{4} - \sqrt{(x-a)(x-b)}\Big|_b^1 \\
&\quad + \left(\frac{a+b}{2}-1\right)\log\left|\frac{2x-a-b+2\sqrt{(x-a)(x-b)}}{4}\right|\Bigg|_b^1 \\
&= \frac{b-a}{2}\log\frac{b-a}{4} + (a+b-2)\log\frac{\sqrt{1-a}+\sqrt{1-b}}{2} \\
&\quad + \sqrt{1-a}\,\sqrt{1-b}.
\end{aligned}
\tag{3.24}
$$

The functionals F_1 and F_2 in (3.19) can be expressed as

$$F_1([a, b]) = \log \frac{b-a}{4} - \frac{1}{2(\alpha - a)} \left\{ f_1 - f_2 - f_3 + \frac{a+b}{2} \log \frac{q}{p} + c_1 \right\}$$

$$F_2([a, b]) = \log \frac{b-a}{4} - \frac{1}{2(b-\alpha)} \left\{ -f_1 - f_2 + f_3 - \frac{a+b}{2} \log \frac{q}{p} + c_2 \right\}.$$

From the formulas for f_1, f_2, and f_3 in (3.22), (3.23), and (3.24) we find

$$\frac{\partial f_1}{\partial a} + \frac{\partial f_1}{\partial b} = -2 \log \frac{\sqrt{a} + \sqrt{b}}{2},$$

$$\frac{\partial f_1}{\partial a} - \frac{\partial f_1}{\partial b} = -\log \frac{b-a}{4} - 1 + \frac{\sqrt{b} - \sqrt{a}}{\sqrt{a} + \sqrt{b}},$$

$$\frac{\partial f_2}{\partial a} + \frac{\partial f_2}{\partial b} = 0,$$

$$\frac{\partial f_2}{\partial a} - \frac{\partial f_2}{\partial b} = 2 \log \frac{b-a}{4} + 2,$$

$$\frac{\partial f_3}{\partial a} + \frac{\partial f_3}{\partial b} = 2 \log \frac{\sqrt{1-a} + \sqrt{1-b}}{2},$$

$$\frac{\partial f_3}{\partial a} - \frac{\partial f_3}{\partial b} = -\log \frac{b-a}{4} - 1 + \frac{\sqrt{1-a} - \sqrt{1-b}}{\sqrt{1-a} + \sqrt{1-b}}.$$

Substituting these expressions in the system (3.20) we get that A, B satisfy the system of equations

$$\beta - \alpha = -\sqrt{AB} + \sqrt{(1-A)(1-B)}, \tag{3.25}$$

$$1 = \frac{\sqrt{1-A} + \sqrt{1-B}}{\sqrt{B} - \sqrt{A}} \cdot \frac{\sqrt{p}}{\sqrt{q}}. \tag{3.26}$$

Observe that (3.25) and (3.26) (after rationalization of the numerator and the denominator of (3.26)), are similar to (3.11) and (3.12), except for the sign of $\sqrt{AB}$ in (3.25) and the sign of $\sqrt{1-B}$ in (3.26), So, the same reasoning leads to the formulas

$$\sqrt{A} = \sqrt{q\alpha} - \sqrt{p\beta}, \qquad \sqrt{B} = \sqrt{q\alpha} + \sqrt{p\beta},$$

which proves the lemma. ∎

Proof of Theorem 2. In Theorem 3 we verify that if $\alpha < p$, then $\mu_{\alpha, p} \leqslant \sigma$, which implies that $\mu_{\alpha, p} = \lambda_{\alpha, p}$. Therefore, by Lemma 4(a) we obtain part (a) of the theorem. Lemma 5 proves part (b), and finally part (c) follows from Theorem 1(a) and 1(b). ∎

4. THE DENSITY OF $\lambda_{\alpha, p}$

Proof of Theorem 3. We observe that part (c) follows from part (a) and Theorem 1(a). Thus, we proceed with the proof of parts (a) and (b) only.

Case $0 < \alpha < p$. In this case we find the measure $\mu_{\alpha, p}$ and observe that it satisfies the constraint $\mu_{\alpha, p} \leqslant \sigma$, which shows that $\lambda_{\alpha, p} = \mu_{\alpha, p}$. Theorem 2(a) asserts that $S_{\mu_{\alpha, p}} = [A, B]$, where A, B are given by (1.9). Since $\mu_{\alpha, p}$ is bounded near the endpoints (we could apply [10, Theorem IV.4.9]), we are led to consider the following boundary value problem, with solution bounded at both end points:

$$\text{p.v.} \frac{1}{\pi} \int_A^B \frac{\phi(t)}{t - x} \, dt = -Q'_{\alpha, p}(x) \qquad \text{on} \qquad [A, B].$$

According to [5, (42.31)] we have

$$\phi(t) = \sqrt{(t - A)(B - t)} \; \text{p.v.} \frac{1}{\pi} \int_A^B \frac{Q'_{\alpha, p}(\tau)}{\sqrt{(\tau - A)(B - \tau}} \frac{d\tau}{\tau - t}, \tag{4.1}$$

where

$$Q'_{\alpha, p}(\tau) = \frac{1}{2\alpha} \left\{ \log \frac{\tau}{1 - \tau} + \log \frac{q}{p} \right\}$$

provided

$$\int_A^B \frac{Q'_{\alpha, p}(\tau)}{\sqrt{(\tau - A)(B - \tau)}} \, d\tau = 0.$$

The last condition can be easily verified for our choice of A and B from which it also follows that $(1/\pi) \int \phi(t) \, dt = 1$. We shall now compute a formula for $\phi(t)$ and in so doing we shall verify that $\mu := (1/\pi) \, \phi(t) \, dt$ is positive on $[A, B]$, so that μ is a probability measure.

To compute the p.v. integral in (4.1) we first observe that since the equilibrium potential $U^{\mu[A,B]}(t) = \text{const.}$ on $[A, B]$, its derivative is zero on $[A, B]$, i.e.,

$$\text{p.v.}\ \frac{1}{\pi}\int_A^B \frac{1}{\tau - t}\frac{d\tau}{\sqrt{(\tau - A)(B-\tau)}} = 0. \tag{4.2}$$

Therefore,

$$\frac{2\alpha\phi(t)}{\sqrt{(t-A)(B-t)}} = \text{p.v.}\ \frac{1}{\pi}\int_A^B \frac{\log\tau - \log(1-\tau)}{\tau - t}\frac{d\tau}{\sqrt{(\tau-A)(B-\tau)}} =: I_1 + I_2. \tag{4.3}$$

In [6] it was shown that

$$\begin{aligned}\text{p.v.}\ &\frac{1}{\pi}\int_a^1 \frac{\log s}{s-t}\frac{ds}{\sqrt{(s-a)(1-s)}}\\ &= \frac{2}{\sqrt{(1-t)(t-a)}}\left(\text{arc tan}\sqrt{\frac{1-t}{t-a}} - \text{arc tan}\sqrt{\frac{(1-t)\,a}{t-a}}\right).\end{aligned} \tag{4.4}$$

Using this formula after the change of variable $s = \tau/B$ in the first integral of (4.3) we obtain with $a = A/B$ (we also use (4.2))

$$\begin{aligned}I_1 &= \text{p.v.}\ \frac{1}{\pi}\int_A^B \frac{\log\tau}{\tau - t}\frac{d\tau}{\sqrt{(\tau-A)(B-\tau)}}\\ &= \frac{1}{B}\,\text{p.v.}\ \frac{1}{\pi}\int_a^1 \frac{\log s}{s - t/B}\frac{ds}{\sqrt{(s-a)(1-s)}}\\ &= \frac{2}{\sqrt{(t-A)(B-t)}}\left(\text{arc tan}\sqrt{\frac{B-t}{t-A}} - \text{arc tan}\sqrt{\frac{A(B-t)}{B(t-A)}}\right).\end{aligned} \tag{4.5}$$

Similarly, after the change $s = (1-\tau)/(1-A)$ with $a = (1-B)/(1-A)$ we get

$$\begin{aligned}I_2 &= -\text{p.v.}\ \frac{1}{\pi}\int_A^B \frac{\log(1-\tau)}{\tau - t}\frac{d\tau}{\sqrt{(\tau-A)(B-\tau)}}\\ &= \frac{1}{1-A}\,\text{p.v.}\ \frac{1}{\pi}\int_a^1 \frac{\log s}{s-(1-t)/(1-A)}\frac{ds}{\sqrt{(s-a)(1-s)}}\\ &= \frac{2}{\sqrt{(t-A)(B-t)}}\left(\text{arc tan}\sqrt{\frac{t-A}{B-t}} - \text{arc tan}\sqrt{\frac{(1-B)(t-A)}{(1-A)(B-t)}}\right).\end{aligned} \tag{4.6}$$

Observe that the two integrals are positive (in particular $\phi(t)$ is non-negative on $[A, B]$). After we add (4.5) and (4.6), and use the formula $\arctan x + \arctan(1/x) = \pi/2$ we see that the resulting measure μ is positive and given by the right-hand side of formula (1.10). Moreover, ϕ is bound above by $1/\alpha$, i.e., $\mu_{\alpha, p} \leqslant \sigma$, therefore $\mu_{\alpha, p} = \lambda_{\alpha, p}$ and we obtain part (a) of Theorem 3.

(b) *Case* $p < \alpha < 1 - p$. The line of proof of this part is somewhat similar to the one in part (a), so we will give just a sketch. In this case we know that $\lambda_{\alpha, p} = m_1$ on the interval $[0, A]$. Set $\lambda_1 := (\alpha/(\alpha - A))(\lambda_{\alpha, p} - m_1)$ (this is the normalized restriction of $\lambda_{\alpha, p}$ on $[A, B]$ so that $\|\lambda_1\| = 1$). Then λ_1 is a solution to an unconstrained weighted energy problem on $[A, B]$ with external field (see (3.17))

$$
\begin{aligned}
Q_{1, A}(x) &= \frac{1}{2(\alpha - A)} \left(U^{m_1}(x) - U^{m_2}(x) - U^{m_3}(x) + x \log \frac{q}{p} + 1 - \log q \right) \\
&= \frac{1}{2(\alpha - A)} \Big\{ -x \log x + (1 - x) \log(1 - x) \\
&\qquad + 2(x - A) \log(x - A) + x \log \frac{q}{p} + 2A - \log q \Big\}.
\end{aligned} \tag{4.7}
$$

In a similar fashion we obtain the formula for the density ϕ_1 of λ_1 (see (4.1))

$$
\phi_1(t) = \sqrt{(t - A)(B - t)}\ \text{p.v.} \frac{1}{\pi} \int_A^B \frac{(Q_{1, A}(\tau))'}{\sqrt{(\tau - A)(B - \tau)}} \frac{d\tau}{\tau - t}, \tag{4.8}
$$

where we find that

$$
(Q_{1, A}(\tau))' = \frac{1}{2(\alpha - A)} \left\{ 2 \log(\tau - A) - \log \tau - \log(1 - \tau) + \log \frac{q}{p} \right\}.
$$

Again using (4.2) one obtains

$$
\frac{2(\alpha - A)\, \phi_1(t)}{\sqrt{(t - A)(B - t)}} = I_3 - I_1 + I_2,
$$

where I_1 and I_2 are already computed in (4.5) and (4.6), and I_3 can be found similarly from the formula (4.4) using the change of variable $s = (\tau - A)/(B - A)$ (now $a = 0$)

$$I_3 := \text{p.v.}\,\frac{1}{\pi}\int_A^B \frac{2\log(\tau-A)}{\tau-t}\,\frac{d\tau}{\sqrt{(\tau-A)(B-\tau)}}$$
$$=\frac{2}{B-A}\,\text{p.v.}\,\frac{1}{\pi}\int_0^1 \frac{\log s}{s-(t-A)/(B-A)}\,\frac{ds}{\sqrt{s(1-s)}}$$
$$=\frac{4}{\sqrt{(t-A)(B-t)}}\arctan\sqrt{\frac{t-A}{B-t}}. \tag{4.9}$$

Thus we have evaluated all the integrals in (4.8), which yields a formula for $d\lambda_1/dt$. Finally, using this formula and the relation $\lambda_{\alpha,p} = (\alpha - A)\,\lambda_1/\alpha + m_1$ we obtain part (b) of Theorem 3. ▮

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